Math8302 HW 1

Exercises in Chapter 6: 5.12, 6.6, 6.7, 6.11, 16.10, 17.15, 7.3, 12.3, 12.19, 12.20
5 problems graded: 6.11, 16.10, 12.3, 12.19, 12.20, each problem 15 points.
Completion 25 points.

Ex. 6.5.12. Suppose $c \in C_k$ and $\partial_k^C = 0$. Then

$$G_k(c) - F_k(c) = \partial_{k+1}^D H_k(c) + H_{k-1}\partial_k^C(c) = \partial_{k+1}^D H_k(c)$$

so $G_*([c]) = [G_k(c)] = [F_k(c)] = F_*([c])$.

Ex. 6.6.6. If $\bar{c} = \overline{c'}$, then $c - c' \in S_k(A)$, so $c - c' = i_\# a, a \in S_k(A)$. Then

$$\partial_k^{(X,A)}(\bar{c}) = \overline{\partial_k^X(c)} = \overline{\partial_k^X(c' + i_\# a)} = \overline{\partial_k^X(c')} + \overline{i_\# \partial_k^A(a)} = \overline{\partial_k^X(c')}$$

with the last equality coming since $\partial_k^A(a) \in S_{k-1}(A)$.

Ex. 6.6.7. The argument reduces to using the fact $f_\# : S(X) \to S(Y)$ is a chain map.

$$f_\# \partial_k^{(X,A)}(\bar{c}) = f_\#(\overline{\partial_k^X(c)}) = \overline{f_\# \partial_k^X(c)} = \overline{\partial_k^Y f_\#(c)} = \partial_k^{(Y,B)}(\overline{f_\#(c)}) = \partial_k^{(Y,B)} f_\#(\bar{c}).$$

Ex. 6.6.11. (a) There is a unique continuous map from $\Delta_k$ to $P$, so the statement follows.

(b) For $k > 0$, $\partial_k(\sigma_k) = \sum_{i=0}^k (-1)^i \sigma_k F_i = \sum_{i=0}^k (-1)^i \sigma_{k-1}$. This has an even number of terms, evenly split with $\pm 1$ coefficients, when $k$ is odd. When $k$ is even, it has one more term with a positive coefficient. For $k = 0$m $\partial_0 = 0$ by definition.

(c) $H_0(P) = \mathbf{Z}/\mathbf{0} = \mathbf{Z}$. For $k = 2p + 1 > 0$, $ker(\partial_k) = S_k(P) = im(\partial_{k+1})$ so $H_k(P) = 0$. For $k = 2p > 0$, $ker(\partial_k) = 0$ so $H_k(P) = 0$.

Ex. 6.16.10. We compute

$$\partial_{i+1}^X H_i^X(\sigma) = \partial_{i+1}^X(\sigma \times id)_\# H_i^{\Delta_i}([e_0, \cdots, e_i]) = (\sigma \times id)_\# \partial_{i+1}^{\Delta_i} H_i^{\Delta_i}([e_0, \cdots, e_i]).$$

Similarly,

$$H_{i-1}^X \partial_i^X(\sigma) = (\sigma \times id)_\# H_{i-1}^{\Delta_i} \partial_i^{\Delta_i}([e_0, \cdots, e_i]).$$

Also,

$$((i_1^X)_\# - (i_0^X)_\#)(\sigma) = (\sigma \times id)_\#((i_1^{\Delta_i})_\# - (i_0^{\Delta_i})_\#)([e_0, \cdots, e_i]).$$

Thus the formula for $X$ follows fro the formula for $\Delta_i$ by composing with the induced map $(\sigma \times id)_\#$. The naturality formula then follows since

$$(f \times id)_\#(\sigma \times id)_\# = (f\sigma \times id)_\#.$$

Ex. 6.17.15. This is done inductively by first defining it for $[e_0, \cdots, e_n]$ by

$$H_n([e_0, \cdots, e_n]) = \tilde{t}.((Id - Sd - H_{n-1}\partial)[e_0, \cdots, e_n]).$$

By Exercise 6.17.2,

$$\partial H_n([e_0,\cdots,e_n]) = (Id - Sd - H_{n-1}\partial)(\partial[e_0,\cdots,e_n]) - \tilde{t}.\partial(Id - Sd - H_{n-1}\partial)([e_0,\cdots,e_n])$$

when $\tilde{t}$ is the barycenter of the simplex.

It suffices to show that the 2nd term vanishes.

By induction, we have $\partial H_{n-1} + H_{n-2}\partial = Id - Sd$. Applying it to $\partial([e_0,\cdots,e_n])$, we have

$$\partial H_{n-1}\partial([e_0,\cdots,e_n]) = (Id-Sd)(\partial[e_0,\cdots,e_n]) - H_{n-2}\partial^2([e_0,\cdots,e_n]) = (Id-Sd)(\partial[e_0,\cdots,e_n]),$$

namely, $\partial(Id - Sd - H_{n-1}\partial)([e_0,\cdots,e_n]) = 0$.

The extension to singular simplices and chains and the formula there then follows the same argument as before.

Ex. 6.7.3. (a) When $x = [\alpha+\beta]$, then $\partial(\alpha) + \partial(\beta) = 0$ since $\alpha+\beta$ is a cycle. But then this means $\partial(\alpha) = -\partial(\beta)$. Since they are equal, they lie in $S_{k-1}(A) \cap S_{k-1}(B) = S_{k-1}(A \cap B)$, where this is considered as a subcomplex of $S_{k-1}(X)$ by inclusion.

(b) The map $\delta$ is induced from the boundary map $\partial$ from part (b) of Exercise 6.7.2. Here $\delta(x) = \partial([\alpha+\beta])$. The long exact sequence in (b) is induced from the short exact sequence of (a). To compute the boundary map, we first pull $\alpha+\beta$ back to $(\alpha,\beta) \in S_k(A) \oplus S_k(B)$. Then we compute $(\partial(\alpha),\partial(\beta)) = (\partial(\alpha), -\partial(\alpha))$ and then pull this $\partial\alpha \in S_{k-1}(A \cap B)$ and get

$$\delta(x) = [\partial(\alpha)].$$

Ex. 6.12.3. (a) Each point $(x,t) \in X \times S^1$ can be connected via a path to a point $(x,1)$. In the quotient space all of the points $(x,1)$ are identified to a single point, so there is a single path component.

(b) In $\Sigma X$, there is a bicollar neighborhood $N$ of $X$ which is the image of $X \times [-1/2, 1/2]$ and this can be used to justify the Mayer-Vietoris sequence where $A = X \times [-1,0]/\sim \subset \Sigma X$ and $B = X \times [0,1]/\sim \subset \Sigma X$. Note that $A, B$ are each contractible, where deformation retracts to $[(x,-1)]$ and $B$ deformation retracts to $[(x,1)]$. Thus $H_k(A) = H_k(B) = 0$ for $k > 0$. Then the portion of the Mayer-Vietoris sequence

$$0 = H_{k+1}(A) \oplus H_{k+1}(B) \longrightarrow H_{k+1}(\Sigma X) \longrightarrow H_k(X) \longrightarrow H_k(A) \oplus H_k(B) = 0$$

implies $H_{k+1}(\Sigma X) \simeq H_k(X)$.

(c) Here we take the portion of the Mayer-Vietoris sequence

$$0 = H_1(A) \oplus H_1(B) \to H_1(\Sigma X) \to H_0(X) \to H_0(A) \oplus H_0(B) = \mathbf{Z} \oplus \mathbf{Z} \to H_0(\Sigma X) = \mathbf{Z} \to 0.$$

The last map is $(a,b) \to a+b$ and so its kernel is isomorphic to $\mathbf{Z}$, Thus the sequence gives a short exact sequence

$$0 \to H_1(\Sigma X) \to H_0(X) \to \mathbf{Z} \to 0$$

which splits to give $H_1(\Sigma X) \oplus \mathbf{Z} \simeq H_0(X)$ since $\mathbf{Z}$ is free abelian.

Ex. 6.12.19. (a) Since $P_{(1)}, Q_{(1)}$ each deformation retract to a wedge of circles, their nonzero homology occurs only in dimensions 1 and 0. The intersection $P_{(1)} \cap Q_{(1)}$ is a circle. From the MV sequence, the terms $H_{k+1}(P_{(1)}) \oplus H_{k+1}(Q_{(1)})$ and $H_k(P_{(1)} \cap Q_{(1)})$ vanish for $k > 1$, giving $H_{k+1}(N) = 0$ for $k > 1$.

(b) The MV sequence gives

$$0 \to H_2(N) \to H_1(S^1) \to H_1(P_{(1)}) \oplus H_1(Q_{(1)}).$$

The map $H_1(S^1) \to H_1(P_{(1)})$ is the map shown above to be multiplication by 2, so is injective. Hence the map $i_1 : H_1(S^1) \to H_1(P_{(1)}) \oplus H_1(Q_{(1)})$ is injective as well. Thus $H_2(N) \simeq ker(i_1) = 0$.

(c) We computed earlier that $\pi_1(P^{(k)}, x)$ is $< a_1, \cdots, a_k | a_1^2 \cdots a_k^2 >$. The abelianization of this group is $(k-1)\mathbf{Z} \oplus \mathbf{Z}_2$. Thus $H_1(P^{(k)})$ is generated by $a_1 + \cdots + a_k, a_2, \cdots, a_h$ with $2(a_1 + \cdots + a_k) = 0$.

(d) Since $P$ is path connected, $H_0(P^{(k)}) \simeq \mathbf{Z}$.

Ex. 6.12.20. (a) Think of the torus coming from a rectangle with identifications on its boundary. Form a small rectangle in the middle and remove it to form $T_{(1)}$. Then on $\pi_1$ the generator of the boundary circle is mapped to the conjugate of the element of $aba^{-1}b^{-1}$. When we abelianize, this becomes the zero map.

(b) The term $T_{(1)}$ is homotopy equivalent to $S^1 \wedge S^1$, so has trivial homology in dimensions $> 1$. From the MV sequence, we get $H_k(T) = 0$ for $k > 2$ and $H_1(T) \simeq H_1(S^1) \simeq \mathbf{Z}$.

(c) $\pi_1(T) \simeq \mathbf{Z} \oplus \mathbf{Z}$, so its abelianization $H_1(T) \simeq \mathbf{Z} \oplus \mathbf{Z}$. By path connectivity, $H_0(T) \simeq \mathbf{Z}$.